# TWO CONJECTURES CONCERNING PARTITIONAL GENERALIZED BINOMIAL COEFFICIENTS

By

Christopher Bingham

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Department of Applied Statistics
University of Minnesota
Minneapolis, Minnesota

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## 1. Introduction and summary

Constantine [2] defined partitional generalized binomial coefficients $\binom{\lambda}{\kappa}$ (his notation was $a_{\lambda,\kappa}$) by means of the identity

$$C_\lambda(I_p + Z)/C_\lambda(I_p) = \sum_{k=0}^{\ell} \sum_{\kappa\in\mathcal{P}} \binom{\lambda}{\kappa} C_\kappa(Z)/C_\kappa(I_p) , \tag{1.1}$$

where the $C_\kappa(Z)$ are zonal polynomials in the $p$ by $p$ symmetric matrix $Z$ [3] and $\mathcal{P}_k = \{\kappa | \kappa = (k_1 \geq k_2 \geq \ldots \geq k_p \geq 0), \sum_{i=1}^{p} k_i = k\}$ is the set of partitions of the integer $k$ into $p$ or fewer parts. Bingham [1] showed that $\binom{\lambda}{\kappa}$ could also be defined by means of the identity

$$C_\lambda(Z)\mathrm{etr}(Z)/k! = \sum_{\ell=k}^{\infty} \sum_{\lambda\in\mathcal{P}_\ell} \binom{\lambda}{\kappa} C_\lambda(Z)/\ell! , \tag{1.2}$$

where $\mathrm{etr}(Z) = \exp(\mathrm{tr}\, Z)$. Let $P = P(Z)$ be a symmetric homogenous polynomial of degree $k$ in the latent roots of $Z$. Then by analogy with (1.2) (but not with (1.1)), Bingham defined coefficients $\binom{\lambda}{P}$ by

$$P(Z)\mathrm{etr}(Z)/k! = \sum_{\ell=k}^{\infty} \sum_{\lambda\in\mathcal{P}_\ell} \binom{\lambda}{P} C_\lambda(Z)/\ell! . \tag{1.3}$$

He gave explicit expressions for $\binom{\lambda}{P}$ for all monomials in $s_j = \mathrm{tr}\, Z^j$ of degree 5 or less. From these one could find expressions for all $\binom{\lambda}{\kappa}$, $\kappa \in \mathcal{P}_k$, $k \leq 5$ using the known relationship between the $C_\kappa(Z)$ and the monomials in $s_j$ (in [3], e.g.).

Based on examination of the formulas in [1] and considering other results derived there, two conjectures concerning the form of $\binom{\lambda}{\kappa}$ for arbitrary $\kappa$ are made in Section 3. If true, these conjectures provide the bases for two algorithms:

1. An algorithm for determining the explicit form of $\binom{\lambda}{\varkappa}$ for all $\varkappa \in \mathcal{P}_k$; and

2. An algorithm for determining the explicit form of $C_\varkappa(Z)$ all $\varkappa \in \mathcal{P}_k$ in terms of the monomials in $s_j$.

Based on these conjectures, expressions for $\binom{\lambda}{P}$ have been determined for all monomials P in $s^j$ of degree $k \le 7$. The expressions found for $C_\varkappa(Z)$ based on the conjectures are correct for all $\varkappa \in \mathcal{P}_k$, $k \le 8$.

## 2. Notation

Throughout we assume $\lambda = (\ell_1,\ldots,\ell_p) \in \mathcal{P}_\ell$, $\varkappa = (k_1,\ldots,k_p) \in \mathcal{P}_k$, and $\rho = (r_1,\ldots,r_p) \in \mathcal{P}_r$. Let $n_k$ be the number of partions in $\mathcal{P}_k$ and $N_k = \sum_{j=1}^k n_j$.

Monomials in $s_j$ will be denoted by

$$\widetilde{C}_\varkappa(Z) = \Pi_{i=1}^p s_{k_i} \ , \quad \text{with} \quad s_0 = 1 \ . \tag{2.1}$$

Define column vectors of length $n_k$ of polynomials in Z by

$$\underset{\sim}{C}_k = [C_\varkappa]_{\varkappa\in\mathcal{P}_k} \ , \tag{2.2}$$

and

$$\underset{\sim}{\widetilde{C}}_k = [\widetilde{C}_\varkappa]_{\varkappa\in\mathcal{P}_k} \ . \tag{2.3}$$

Vectors $\underset{\sim}{C}_k$ and $\underset{\sim}{\widetilde{C}}_k$ are related by

$$\underset{\sim}{C}_k = U_k \, \underset{\sim}{\widetilde{C}}_k \tag{2.4}$$

where $U_k$ is a $n_k$ by $n_k$ matrix. The matrices $U_k$ are known at least up to $k = 11$.

Define $n_k$ by $n_\ell$ matrices $R_{k\ell}$ and $\tilde{R}_{k\ell}$ by

$$R_{k\ell} = [\binom{\lambda}{\varkappa}]_{\varkappa \in \mathcal{P}_k,\ \lambda \in \mathcal{P}_\ell} \tag{2.5}$$

and

$$\tilde{R}_{k\ell} = [\binom{\lambda}{\tilde{C}_\varkappa}]_{\varkappa \in \mathcal{P}_k,\ \lambda \in \mathcal{P}_\ell} . \tag{2.6}$$

Then, by (1.1) or (1.2)

$$R_{kk} = I_{n_k}, \text{ and } R_{k\ell} = 0, \quad k > \ell . \tag{2.7}$$

By (1.2) and (2.4)

$$\tilde{R}_{kk} = U_k^{-1}, \quad \tilde{R}_{k\ell} = U_k^{-1} \cdot R_{k\ell}, \text{ and } \tilde{R}_{k\ell} = 0, \quad k > \ell. \tag{2.8}$$

Define the $N_k$ by $N_k$ matrices $R^{(k)}$ and $\tilde{R}^{(k)}$ by

$$R^{(k)} = [R_{r,\ell}]_{r,\ell=1}^{k} \quad \text{and} \quad \tilde{R}^{(k)} = [\tilde{R}_{r,\ell}]_{r,\ell=1}^{k} . \tag{2.9}$$

Bingham [1] introduced certain partitional coefficients $d_k(\lambda)$ by

$$d_0(\lambda) = 1, \quad d_1(\lambda) = \ell, \quad d_k(\lambda) = \Sigma_{i=1}^{p} \Sigma_{j=1}^{\ell_i} (j - \frac{i-1}{2})^{k-1}, \ k \geq 2. \tag{2.10}$$

Clearly $d_k(\lambda)$ can be expressed as a polynomial of degree $k$ in $\ell_1, \ldots, \ell_p$. Define monomials of _weight_ _k_ in the $d_r(\lambda)$'s by

$$d_{\varkappa}(\lambda) = \Pi_{i=1}^{p} d_{k_i}(\lambda) \; . \tag{2.11}$$

Finally define $N_k$ by $N_k$ matrices

$$D^{(k)} = [D_{r,\ell}]_{r,\ell=1}^{k} \tag{2.12}$$

where the $n_r$ by $n_\ell$ matrices $D_{r,\ell}$ are defined

$$D_{r,\ell} = [d_\rho(\lambda)]_{\rho\in\mathcal{P}_r,\ \lambda\in\mathcal{P}_\ell} \; . \tag{2.13}$$

## 3. Conjectured results.

Examination of the expressions for $\binom{\lambda}{\tilde{C}_\varkappa}$ presented in [1] for all $\varkappa \in \mathcal{P}_k$, $k \le 5$ reveals two facts:

1. The coefficients $\binom{\lambda}{\tilde{C}_\varkappa}$ are expressible as linear combinations of partitional monomials $d_\rho(\lambda)$, $\rho \in \mathcal{P}_r$, $r \le k$; and

2. The *only* partitional monomial of weight $k$ entering the expression for $\binom{\lambda}{\tilde{C}_\varkappa}$ is $g_\varkappa \, d_\varkappa(\lambda)/k!$ where

$$g_\varkappa = \prod_{k_i \ge 1} (k_i) \; . \tag{3.1}$$

Both assertions are conjectured to hold for all $k$. The first can be made more precise as:

*Conjecture 1*:

$$\tilde{R}^{(k)} = \tilde{A}^{(k)} D^{(k)} , \tag{3.2}$$

where

$$\tilde{A}^{(k)} = [\tilde{A}_{r,\ell}]_{r,\ell=1}^{k}, \; \tilde{A}_{r,\ell} = 0, \; r < \ell \text{ where } \tilde{A}_{r,\ell} \text{ is } n_r \times n_\ell \, . \tag{3.3}$$

Clearly equivalent to Conjecture 1 in view of (2.8) is

*Conjecture 1'*:

$$R^{(k)} = A^{(k)} D^{(k)}, \tag{3.4}$$

where

$$A^{(k)} = [A_{r,\ell}]_{r,\ell=1}^{k},\ A_{r,\ell} = 0,\ r < \ell \text{ where } A_{r,\ell} \text{ is } n_r \times n_\ell . \tag{3.5}$$

Clearly from (3.2) and (2.8)

$$A_{\ell,r} = U_\ell \tilde{A}_{\ell,r} . \tag{3.6}$$

The second assertion above is equivalent to:

<u>Conjecture 2</u>:

$$\tilde{A}_{kk} = (k!)^{-1} \operatorname{diag}[g_\varkappa] . \tag{3.7}$$

A direct consequence of Conjecture 2 and (3.6) is:

<u>Conjecture 2'</u>:

$$A_{kk} = (k!)^{-1} U_k \operatorname{diag}[g_\varkappa] . \tag{3.8}$$

The author has been unable to prove these conjectures. However, their plausibility is strengthened by various facts:

(a) The conjectures are true for $k \le 5$.

(b) Theorem 2 of [1] states in part that, if $P(Z)$ is a homogeneous symmetric polynomial of degree $k$, then

$$\binom{\lambda}{s_1^r P} = \frac{(\ell-k)(\ell-k-1)\ .\ .\ .\ (\ell-k-r+1)}{(k+r)(k+r-1)\ .\ .\ .\ (k+1)} \binom{\lambda}{P} . \tag{3.9}$$

Since $d_1(\lambda) = \ell$, this clearly implies that if Conjecture 1 or Conjecture 2 is true for $\tilde{C}_k$ it is also true for $\tilde{C}_{(\varkappa,1^r)} = s_1^r \tilde{C}_\varkappa$.

(c) It was proved in [1] that if

$$\begin{aligned} P_{r,t}(Z) &= r!\ (\text{term of degree } r \text{ in } [\textstyle\sum_{j=1}^{\infty} s_j/j]^t/t!) \\ &= r! \sum_{\substack{k_1+2k_2+3k_3+\ldots=r \\ k_1+k_2+k_3+\ldots=t}} \frac{(s_1)^{k_1}(s_2/2)^{k_2}(s_3/3)^{k_3}\ .\ .\ .}{k_1!\ k_2!\ k_3!\ .\ .\ .} \end{aligned} \tag{3.10}$$

then

$$\binom{\lambda}{p_{r,t}} = \sum_{\rho\in\mathcal{P}_r} f_{r-t}(\rho)\binom{\lambda}{\rho}, \tag{3.11}$$

where $f_{r-t}(\rho)$ is defined by

$$\sum_{t=0}^{r} f_{r-t}(\rho)x^t = (x)_\rho \equiv \prod_{i=1}^{p}\prod_{j=1}^{r_i}(x + j - \tfrac{1}{2}(i-1)). \tag{3.12}$$

It was further shown in [1] that

$$(n(1-z))_\lambda/(n)_\lambda = \sum_{t=0}^{\ell}\sum_{k=0}^{\infty}(-1)^{t+k}z^t n^{-k} \times \\ (\sum_{\rho\in\mathcal{P}_{t+k}} f_k(\rho)\binom{\lambda}{\rho} + [\text{linear combination of } \binom{\lambda}{\rho'},\ \rho'\in\mathcal{P}_r,\quad r < t+k]). \tag{3.13}$$

Another expression for $(n(1-z))_\lambda/(n)_\lambda$ in [1] is equivalent to

$$(n(1-z))_\lambda/(n)_\lambda = \exp\{-\sum_{k=0}^{\infty}(-1)^k n^{-k} d_{k+1}(\lambda)\, h_k(z)\}, \tag{3.14}$$

where

$$h_k(Z) = \sum_{r=1}^{\infty}\frac{(k+r-1)!}{k!\ r!}z^r = z + \tfrac{1}{2}(k+1)z^2 + \dots\ . \tag{3.15}$$

After straightforward rearrangments of (3.14) we find that

$$(n(1-z))_\lambda/(n)_\lambda = 1 + \sum_{t=1}^{\infty}(-1)^t\sum_{k=0}^{\infty}(-1)^k n^{-k} \\ \times \Big(\sum_{\substack{k_1+2k_2+3k_3+\dots=k+t \\ k_1+k_2+k_3+\dots=t}} \frac{d_1^{k_1}(\lambda)d_2^{k_2}(\lambda)d_3^{k_3}(\lambda)\dots}{k_1!\ k_2!\ k_3!\ \dots}\, h_0^{k_1}(z)\, h_1^{k_2}(z)\, h_2^{k_3}(z)\dots\Big). \tag{3.16}$$

Since (3.15) implies that $h_0^{k_1}(z)\, h_1^{k_2}(z)\, h_2^{k_3}(z) \ldots = z^t + O(z^{t+1})$, if $\sum k_j = t$, (3.16) demonstrates that the term of highest weight in the $d_j(\lambda)$'s in the coefficient of $z^t n^{-k}$ has weight $k + t$ and is

$$(-1)^{t+k} \mathcal{D}_{k+t,k} \equiv (-1)^{t+k} \sum_{\substack{k_1+2k_2+3k_3+\ldots=k+t \\ k_1+k_2+k_3+\ldots=t}} \frac{d_1^{k_1}\, d_2^{k_2}\, d_3^{k_3} \ldots}{k_1!\, k_2!\, k_3! \ldots} \, . \tag{3.17}$$

Thus, using (3.11) and (3.13), this implies that

$$\begin{aligned} \binom{\lambda}{p_{t+k,k}} = \mathcal{D}_{k+t,k} &+ [\text{linear combination of lower order } \binom{\lambda}{\rho'}] \\ &+ [\text{terms of weight} < k+t \text{ in the } d_j(\lambda)\text{'s}]. \end{aligned} \tag{3.18}$$

If Conjecture 1 is true for degrees $< r$, then (3.18) implies:

(i) $\binom{\lambda}{p_{r,t}}$ is a linear combination of monomials in $d_j(\lambda)$ of weight not greater than $r$; and

(ii) the term of highest weight in $\binom{\lambda}{p_{r,t}}$ is $\mathcal{D}_{r,t}$. It will be observed that (i) would be implied by Conjecture 1 while Conjecture 2 predicts (ii). This further enhances the plausibility of the conjectures.

## 4. Using the conjectures to find $A^{(k)}$ and $U_k$

If we assume the truth of Conjecture 1', (3.4) implies

$$A^{(k)} = R^{(k)} [D^{(k)}]^{-1} , \tag{4.1}$$

provided $D^{(k)}$ is invertible. Partition $[D^{(k)}]^{-1}$ into submatrices $D^{r,\ell}_{(k)}$ of order $n_r$ by $n_\ell$, $r \le k$, $\ell \le k$. Then (4.1) and (2.7) together imply that

$$[A_{k1}, A_{k2}, \ldots, A_{kk}] = [D^{k1}_{(k)}, D^{k2}_{(k)}, \ldots, D^{kk}_{(k)}] . \tag{4.2}$$

This provides the basis for an algorithm for finding all the non-zero elements of $A^{(k)}$ and hence expressions for all partitional generalized binomial coefficients of order $\le k$, by successive inversion of $D^{(1)}$, $D^{(2)}, \ldots$ . The computation can be made easier by means of the well known method of using $[D^{(r-1)}]^{-1}$ to compute $[D^{(r)}]^{-1}$, the largest matrix requiring direct inversion at any step being of order $n_r$ by $n_r$. It is clear from the discussions in Section 2 that if, in the definition of $D^{(k)}$, we replace $d_1^{k_1} d_2^{k_2} d_3^{k_3} \ldots$ , of weight $r$, by $(d_1-r+k_1)(d_1-r+k_1-1) \ldots (d_1-r+1) d_2^{k_2} d_3^{k_3} \ldots$ , then we can restate our conjectures using the redefined $D^{(k)}$. The matrix $A^{(k)}$ will be different but will satisfy the same properties as there discussed. The resulting modified form of $A^{(k)}$ has a somewhat simpler structure than as originally defined in Section 2. Moreover, there is some evidence of greater numerical stability in carrying out the algorithm described above.

From (3.6), $\tilde{A}^{(k)}$ (with or without the definition modified as in the preceding paragraph) can be determined from $A^{(k)}$ once the $U_r$ are known for $r \le k$. In the Appendix are entries for the (modified) matrix $\tilde{A}^{(7)}$ computed using the algorithm discussed above.

If we assume the truth of Conjecture 2, then (4.2) and (3.8) imply

$$U_k = k!\ \mathrm{diag}[g_\varkappa^{-1}]D_{(k)}^{kk} \ . \tag{4.3}$$

Thus the algorithm described above also can be used to determine expressions for the zonal polynomials in terms of the $\tilde{C}_\varkappa$. Numerical computations verify (4.3) for $k \le 8$, thus strengthening the plausibility of the Conjectures.

Appendix.

Table 1 contains the matrix of coefficients for expressing $(k!/g_\varkappa)\binom{\lambda}{\tilde{C}_\varkappa}$ for all $\varkappa \in \mathcal{P}_k$, $k \le 7$ in terms of the partitional monomials $d_\rho(\lambda)$ as modified in Section 4. The table also contains values of $g_\varkappa$. Thus for example we find

$$\begin{aligned}\binom{\lambda}{s_3 s_2} = {} & (1/20)(d_3(\lambda)d_2(\lambda) - (1/4)(d_1(\lambda)-3)(d_1(\lambda)-2)d_2(\lambda) \\ & - (1/2)d_2^2(\lambda) - 4d_4(\lambda) + 2(d_1(\lambda)-2)d_2(\lambda) \\ & + (9/2)d_3(\lambda) - (5/8)(d_1(\lambda)-1)d_1(\lambda) + (1/4)d_2(\lambda)\ ) \ .\end{aligned}$$

Table 1 was computed using KRONOS APL on a CDC 6400. Each of the matrices $D^{(k)}$, $k = 1,\ldots, 7$ was inverted using Gauss-Jordan elimination with pivoting. After multiplying the floating point values so obtained by the indicated denominators and the factor $k!/g_\varkappa$, the resulting values were all within $10^{-6}$ of being integers. In view of the numerical check of (4.3) for all $k \le 8$ (to equivalent accuracy), I feel the truth of the conjectures, at least up to $k = 8$, is extremely likely and would not hesitate to recommend the use of Table 1.

| $\varkappa$ | $k!/g_\varkappa$ | $d_1$ | $d_2$ | $d_1(d_1-1)$ | $d_3$ | $d_2(d_1-2)$ | $d_1(d_1-1)(d_1-2)$ | $d_4$ | $d_3(d_1-3)$ | $d_2^2$ | $d_2(d_1-2)(d_1-3)$ | $d_1(d_1-1)(d_1-2)(d_1-3)$ | $d_5$ | $d_4(d_1-4)$ | $d_3d_2$ | $d_3(d_1-3)(d_1-4)$ | $d_2^2(d_1-4)$ | $d_2(d_1-2)(d_1-3)(d_1-4)$ | $d_1(d_1-1)(d_1-2)(d_1-3)(d_1-4)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | | | | | | | | | | | | | | | | | |
| 2 | 1 | 0 | 1 | | | | | | | | | | | | | | | | |
| $1^2$ | 2 | 0 | 0 | 1 | | | | | | | | | | | | | | | |
| 3 | 2 | 0 | -1/2 | -1/4 | 1 | | | | | | | | | | | | | | |
| 21 | 3 | 0 | 0 | 0 | 0 | 1 | | | | | | | | | | | | | |
| $1^3$ | 6 | 0 | 0 | 0 | 0 | 0 | 1 | | | | | | | | | | | | |
| 4 | 6 | 0 | 0 | 1/4 | -3/2 | -1 | 0 | 1 | | | | | | | | | | | |
| 31 | 8 | 0 | 1/4 | 1/2 | 0 | -1/2 | -1/4 | 0 | 1 | | | | | | | | | | |
| $2^2$ | 6 | 0 | 1 | 1/2 | -3 | 0 | 0 | 0 | 0 | 1 | | | | | | | | | |
| $21^2$ | 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | | | | | | |
| $1^4$ | 24 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | | | | | |
| 5 | 24 | 0 | -1/4 | -3/4 | 13/4 | 9/4 | 5/24 | -3 | -3/2 | -1 | 0 | 0 | 1 | | | | | | |
| 41 | 30 | 0 | 0 | -1/2 | 3/2 | 1 | 1/4 | 0 | -3/2 | 0 | -1 | 0 | 0 | 1 | | | | | |
| 32 | 20 | 0 | 1/4 | -5/8 | 9/2 | 2 | 0 | -4 | 0 | -1/2 | -1/4 | 0 | 0 | 0 | 1 | | | | |
| $31^2$ | 40 | 0 | -1 | -1/2 | 0 | 1 | 1/2 | 0 | 0 | 0 | -1/2 | -1/4 | 0 | 0 | 0 | 1 | | | |
| $2^21$ | 30 | 0 | -2 | -1 | 3 | 1 | 1/2 | 0 | -3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | |
| $21^3$ | 60 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | |
| $1^5$ | 120 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 6 | 120 | 0 | -5/4 | 5/2 | -47/4 | -15/2 | -2/3 | 53/4 | 6 | 15/4 | 3/2 | 0 | -5 | -2 | -3 | 0 | 0 | 0 | 0 |
| 51 | 144 | 0 | 3/4 | 9/4 | -26/4 | -19/4 | -7/6 | 3 | 19/4 | 1 | 9/4 | 5/24 | 0 | -3 | 0 | -3/2 | -1 | 0 | 0 |
| 42 | 90 | 0 | 0 | 17/8 | -35/4 | -13/2 | -2/3 | 12 | 6 | 2 | 1/4 | 0 | -5 | 0 | -3/2 | 0 | -1 | 0 | 0 |
| $41^2$ | 180 | 0 | 0 | 3/2 | -3 | -2 | -1 | 0 | 3 | 0 | 2 | 1/4 | 0 | 0 | 0 | -3/2 | 0 | -1 | 0 |
| $3^2$ | 80 | 0 | -1/2 | 7/4 | -33/4 | -6 | -1/2 | 11 | 3 | 13/4 | 1/4 | 1/16 | -5 | 0 | -1 | -1/2 | 0 | 0 | 0 |
| 321 | 120 | 0 | -3/4 | 15/8 | -9 | -15/4 | -5/8 | 4 | 9/2 | 1/2 | 9/4 | 0 | 0 | -4 | 0 | 0 | -1/2 | -1/4 | 0 |
| $31^3$ | 240 | 0 | 3 | 3/2 | 0 | -3 | -3/2 | 0 | 0 | 0 | 3/2 | 1/2 | 0 | 0 | 0 | 0 | 0 | -1/2 | -1/4 |
| $2^3$ | 90 | 0 | -3/2 | 9/4 | -18 | -6 | 0 | 20 | 0 | 3 | 3/2 | 0 | 0 | 0 | -9 | 0 | 0 | 0 | 0 |
| $2^21^2$ | 180 | 0 | 6 | 3 | -6 | -4 | -2 | 0 | 6 | 0 | 1 | 1/2 | 0 | 0 | 0 | -3 | 0 | 0 | 0 |
| $21^4$ | 360 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $1^6$ | 720 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 720 | 0 | 63/8 | -185/16 | 733/16 | 279/8 | 4 | -539/8 | -269/8 | -273/16 | -121/16 | -49/192 | 35 | 25/2 | 69/4 | 21/8 | 7/2 | 0 | 0 |
| 61 | 840 | 0 | 5 | -10 | 141/4 | 85/4 | 9/2 | -53/2 | -95/4 | -15/2 | -21/2 | -2/3 | 5 | 61/4 | 3 | 6 | 15/4 | 3/2 | 0 |
| 52 | 504 | 0 | 13/2 | -67/8 | 151/4 | 24 | 59/24 | -105/2 | -51/2 | -47/4 | -23/4 | 0 | 25 | 10 | 61/4 | 0 | 9/4 | 5/24 | 0 |
| $51^2$ | 1008 | 0 | -3 | -9 | -39/2 | 15 | 23/4 | -6 | -16 | -2 | -37/4 | -19/12 | 0 | 6 | 0 | 25/4 | 2 | 9/4 | 5/24 |
| 43 | 420 | 0 | 47/8 | -53/8 | 231/8 | 159/8 | 33/16 | -40 | -39/2 | -12 | -37/8 | -1/16 | 45/2 | 8 | 39/4 | 5/8 | 1/2 | 1/4 | 0 |
| 421 | 630 | 0 | 0 | -17/2 | 105/4 | 39/2 | 33/8 | -24 | -83/4 | -4 | -7 | -2/3 | 5 | 12 | 3/2 | 6 | 3 | 1/4 | 0 |
| $41^3$ | 1260 | 0 | 0 | -6 | 9 | 6 | 9/2 | 0 | -9 | 0 | -6 | -3/2 | 0 | 0 | 0 | 9/2 | 0 | 3 | 1/4 |
| $3^21$ | 560 | 0 | 2 | -7 | 99/4 | 35/2 | 12/4 | -22 | -57/4 | -13/2 | -13/2 | -5/8 | 5 | 11 | 1 | 7/2 | 13/4 | 1/4 | 1/16 |
| $32^2$ | 420 | 0 | 13/4 | -61/8 | 63/2 | 25 | 5/2 | -58 | -45/2 | -8 | -7/4 | -1/8 | 30 | 0 | 23/2 | 5/4 | 4 | 0 | 0 |
| $321^2$ | 840 | 0 | 3 | -15/2 | 27 | 21/2 | 15/4 | -8 | -18 | -1 | -33/4 | -5/8 | 0 | 8 | 0 | 9/2 | 1 | 5/2 | 0 |
| $31^4$ | 1680 | 0 | -12 | -6 | 0 | 12 | 6 | 0 | 0 | 0 | -6 | -3 | 0 | 0 | 0 | 0 | 0 | 2 | 1 |
| $2^31$ | 630 | 0 | 6 | -9 | 54 | 33/2 | 9/4 | -40 | -18 | -6 | -9 | 0 | 0 | 20 | 9 | 0 | 3 | 3/2 | 0 |
| $2^21^3$ | 1260 | 0 | -24 | -12 | 18 | 18 | 9 | 0 | -18 | 0 | -6 | -3 | 0 | 0 | 0 | 9 | 0 | 1 | 1/2 |
| $21^5$ | 2520 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $1^7$ | 5040 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| $\varkappa$ | $d_6$ | $d_5(d_1-5)$ | $d_4d_2$ | $d_4(d_1-4)(d_1-5)$ | $d_3^2$ | $d_3d_2(d_1-5)$ | $d_3(d_1-3)(d_1-4)(d_1-5)$ | $d_2^3$ | $d_2^2(d_1-4)(d_1-5)$ | $d_2(d_1-2)(d_1-3)(d_1-4)(d_1-5)$ | $d_1(d_1-1)(d_1-2)(d_1-3)(d_1-4)(d_1-5)$ | $d_7$ | $d_6(d_1-6)$ | $d_5d_2$ | $d_5(d_1-5)(d_1-6)$ | $d_4d_3$ | $d_4d_2(d_1-6)$ | $d_4(d_1-4)(d_1-5)(d_1-6)$ | $d_3^2(d_1-6)$ | $d_3d_2^2$ | $d_3d_2(d_1-5)(d_1-6)$ | $d_3(d_1-3)(d_1-4)(d_1-5)(d_1-6)$ | $d_2^3(d_1-6)$ | $d_2^2(d_1-4)(d_1-5)(d_1-6)$ | $d_2(d_1-2)(d_1-3)(d_1-4)(d_1-5)(d_1-6)$ | $d_1(d_1-1)(d_1-2)(d_1-3)(d_1-4)(d_1-5)(d_1-6)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 1 | | | | | | | | | | | | | | | | | | | | | | | | |
| 51 | 0 | 1 | | | | | | | | | | | | | | | | | | | | | | | |
| 42 | 0 | 0 | 1 | | | | | | | | | | | | | | | | | | | | | | |
| $41^2$ | 0 | 0 | 0 | 1 | | | | | | | | | | | | | | | | | | | | | |
| $3^2$ | 0 | 0 | 0 | 0 | 1 | | | | | | | | | | | | | | | | | | | | |
| 321 | 0 | 0 | 0 | 0 | 0 | 1 | | | | | | | | | | | | | | | | | | | |
| $31^3$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | | | | | | | | | | | | | | | | |
| $2^3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | | | | | | | | | | | | | | | |
| $2^21^2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | | | | | | | | | | | | | | |
| $21^4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | | | | | | | | | | | | | |
| $1^6$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | | | | | | | | | | | | |
| 7 | -15/2 | -5/2 | -4 | 0 | -9/4 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | | | | | | | | | | | |
| 61 | 0 | -5 | 0 | -2 | 0 | -3 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | | | | | | | | | | |
| 52 | -6 | 0 | -3 | 0 | 0 | -3/2 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 1 | | | | | | | | | | | |
| $51^2$ | 0 | 0 | 0 | -3 | 0 | 0 | -3/2 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 1 | | | | | | | | | | |
| 43 | -6 | 0 | -1/2 | -1/4 | -3/2 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | | | | | | | |
| 421 | 0 | -5 | 0 | 0 | 0 | -3/2 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | | | | | | |
| $41^3$ | 0 | 0 | 0 | 0 | 0 | 0 | -3/2 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | | | | | |
| $3^21$ | 0 | -5 | 0 | 0 | 0 | -1 | -1/2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | | | | |
| $32^2$ | 0 | 0 | -8 | 0 | -3 | 0 | 0 | -1/2 | -1/4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | | | |
| $321^2$ | 0 | 0 | 0 | -4 | 0 | 0 | 0 | 0 | -1/2 | -1/4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | | |
| $31^4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1/2 | -1/4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | | |
| $2^31$ | 0 | 0 | 0 | 0 | 0 | -9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | | |
| $2^21^3$ | 0 | 0 | 0 | 0 | 0 | 0 | -3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | |
| $21^5$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| $1^7$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

<u>Table 1.</u> Matrix of coefficients expressing $(k!/g_\varkappa)(\chi_\varkappa^\lambda)$ in terms of modified monomials in $d_j(\lambda)$.

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